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Type-sequences of modules

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Abstract

Let R be a complete and integral local k-algebra of dimension one, k an algebraically closed field of characteristic zero. In this paper the notion of *type-sequence*, given for rings in Barucci et al. (AMS Mem. 125 (598) (1997) Ch. II,1), is extended to any finitely generated torsion-free R-module of rank 1. A module M, of Cohen–Macaulay type $r_1(M)$, whose type-sequence is $[r_1(M), 1, \ldots, 1]$ is said to have "*minimal type-sequence*", briefly *m.t.s.* The family of *m.t.s.* R-modules, which includes the canonical module, is described by means of value sets, the conductor $c(M)$, the δ-invariant $\delta(M)$ and the C.M. type $r_1(M)$. In the case of rings the *m.t.s.* property is called "*almost Gorenstein*" (see Barucci and Fröberg, J. Algebra 188 (1997) 418–442). Inspired by analogous investigations by Barucci and Fröberg, we study in Section 3 the *m.t.s.* property and the reflexiveness of modules over almost Gorenstein rings. © 2001 Elsevier Science B.V. All rights reserved.

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1.

Let k be an algebraically closed field of characteristic zero. Let $R = k[\![x_1, \ldots, x_n]\!]$ be a complete one-dimensional local k-algebra, with maximal ideal $\mathfrak{m}$. We assume R is an integral domain.

Let $\bar{R} := k[\![t]\!]$ be the integral closure of R in its quotient field $K := k\{\{t\}\}$ and let $v : k\{\{t\}\} \to \mathbb{Z} \cup \infty$ be the canonical valuation, given by the degree in t.

We have the following invariants and numerical sets associated with R:

$\Gamma := \{v(r) \mid r \in R\}$ is called the *value semigroup* of R,

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$\delta := l_R(\bar{R}/R)$ is called the *singularity degree* of R,

$c := \min\{v(r), r \in Ann_R(\bar{R}/R)\}$ is called the *conductor* of R,

$r_1(R) := l_R(R : \mathfrak{m})/R$ is called the *Cohen–Macaulay type* of R,

$e := l_R(\bar{R}/\mathfrak{m}\bar{R})$ is called the *multiplicity* of R.

We shall assume that $e > 1$ and that $t^e \in \mathfrak{m}$ (this is always possible after a suitable change of coordinates).

Analogously, for any fractional R-ideal $M \subseteq K$, we let

$\Gamma(M) := \{v(m) \mid m \in M\}$ be *the value set* of M,

$\delta(M) := l_R(\bar{M}/M)$ (*where* $\bar{M} := M\bar{R}$) be *the δ-invariant* of M,

$c(M) := \min\{x \in \mathbb{Z} \mid t^x\bar{R} \subseteq M\}$ be *the conductor* of M,

$r_1(M) := l_R(M : \mathfrak{m})/M$ be *the Cohen–Macaulay type* of M.

Note that $\Gamma(M)$ is a *Γ-set*, i.e., $\Gamma(M) + \Gamma \subseteq \Gamma(M)$.

Assumption 1.1. For any isomorphism class of finitely generated torsion-free R-modules of rank one we will always choose a representative M such that

$$R \subseteq M \subseteq \bar{R}.$$

(This is always possible: see [4, Lemma 1.1]).

Remark 1.2. Having made that choice we have

$$\delta(M) = l_R(\bar{R}/M) \qquad \text{and} \qquad l_R(M/t^{c(M)}\bar{R}) = c(M) - \delta(M).$$

Notation 1.3. Let $c_0 = 0 < c_1 = 2 < \cdots < c_\delta = c$ be those integers such that $c_j - 1 \notin \Gamma$, $\forall j = 0, \ldots, \delta$. We shall denote $R_j := R + t^{c_j}\bar{R}$.

Remark 1.4. The overrings R_j introduced in Notation 1.3 constitute a chain of rings intermediate between R and $\bar{R}$,

$$\bar{R} = R_0 \supseteq R_1 \supseteq \cdots \supseteq R_j \supseteq \cdots \supseteq R_\delta = R.$$

Clearly, all the inclusions above are proper and we also have $\delta(R_j) = j$.

Because of the choice $R \subseteq M \subseteq \bar{R}$, it is easy to see that there exists a $k \in \{0, \ldots, \delta\}$ such that $c(M) = c_k$ and $M \supseteq R_k$. It follows that M is also an R_k-module.

We now want to extend the definition of type-sequences for rings, given in [1], to modules. So, let M be an R-module. Let

$$\Gamma = \{s_0 = 0 < s_1 = e < \cdots < s_{n-1} < s_n = c, \rightarrow\}$$

be the value set of R (hence $n = c - \delta$). Consider, for all $i = 0,\dots,n$, the ideals $V_i := \{x \in R, v(x) \geq s_i\}$. Obviously, $V_n = t^c\bar{R}$, $V_1 = \mathfrak{m}$, $V_0 = R$ and

$$V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq V_0.$$

Starting from this maximal chain we get the following chain of fractional ideals:

$$M = M{:}V_0 \subseteq M{:}V_1 \subseteq \cdots \subseteq M{:}V_n.$$

Since $V_n = t^c\bar{R}$, we get that $M{:}V_n = t^{c_k - c}\bar{R}$. Now, let

$$r_i = r_i(M) := l_R(M{:}V_i/M{:}V_{i-1}), \quad i = 1,\dots,n.$$

Definition 1.5. The sequence $[r_1,\dots,r_n]$ is called the *type-sequence* of M. We denote this by *t.s.* $(M) := [r_1,\dots,r_n]$.

Note that $r_1 = l_R(M{:}\mathfrak{m}/M)$ is the CM-type of M. Moreover,

Proposition 1.6. *Let $c(M) = c_k$ and let $n = c - \delta$. Then*
(i) $c - c_k + \delta(M) = l_R(t^{c_k - c}\bar{R}/M) = r_1(M) + \sum_{i=2}^{n} r_i(M)$;
(ii) $1 \leq r_i(M) \leq r_1(M)$ $\forall i = 1,\dots,n$;
(iii) $r_1(M) - 1 \leq \delta + \delta(M) - c_k$.

Proof. (i) is immediate.

(ii) Since for all $i = 1,\dots,n$, the element $z := t^{c_k - 1 - s_{i-1}}$ verifies $z \in M{:}V_i$ and $z \notin M{:}V_{i-1}$, we have $r_i \geq 1$. As for the second inequality: set $\mathfrak{b} := V_i$ and $\mathfrak{a} := V_{i-1}$. By definition $l_R(\mathfrak{a}/\mathfrak{b}) = 1$, hence by [6, Satz 2], we obtain that $r_i = l_R(M{:}\mathfrak{b}/M{:}\mathfrak{a}) \leq r_1$.

(iii) This was already noted in [7, Proposition 1.4], but using the fact that $n = c - \delta$ and that $r_i \geq 1$, $\forall i = 2,\dots,n$, the inequality follows easily from equality (i). $\square$

Let M be as above and let $c(M) = c_k$. Since we noted that M is also a module over the ring $R_k := R + t^{c_k}\bar{R}$, it is natural to associate with M its *type-sequence* $[l_1,\dots,l_m]$ as an R_k-module, $m = c_k - k \leq n$. Here $l_i = l_i(M)$, $\forall i = 1,\dots,m$, and l_1 is the Cohen–Macaulay type of M as an R_k-module.

In order to distinguish this type-sequence from the one already seen, we call it the *k-type-sequence* of M, briefly *k-t.s.* (M).

Using Proposition 1.6 and considering M as an R_k-module, we obtain

Proposition 1.7. *Let $c(M) = c_k$ and let $m = c_k - k$. Then*
(i) $\delta(M) = l_1(M) + \sum_{i=2}^{m} l_i(M)$ *and* $1 \leq l_i(M) \leq l_1(M)$, $\forall i = 1,\dots,m$;
(ii) $\delta(M) \leq (c_k - k)l_1(M)$;
(iii) $\delta(M) = (c_k - k)l_1(M) \Leftrightarrow$ *k-t.s.* $(M) = [l_1(M),\dots,l_1(M)]$.

Remark 1.8. Given two fractional ideals N_1, N_2, with $N_2 \subseteq N_1$, we can compute the length of the R-module N_1/N_2 by means of valuations. In fact, it is well known that $l_R(N_1/N_2) = \#[\Gamma(N_1) \setminus \Gamma(N_2)]$.

Notation 1.9. For any $H,K \subseteq \mathbb{Z}$ we denote by

$$H - K{:=}\{x \in \mathbb{Z} \,|\, x + K \subseteq H\} \quad \text{and} \quad H^+{:=}H \cap \mathbb{N},\ H^-{:=}H \setminus H^+.$$

We will now see that each invariant l_i represents the "positive contribution" of the corresponding r_i (see Proposition 1.10(i)). We also find an upper bound for the difference $r_i - l_i$.

Proposition 1.10. *Let $c(M) = c_k$. Let t.s. $(M) = [r_1,\dots,r_m,r_{m+1},\dots,r_n]$, with $n = c - \delta$, and let k-t.s. $(M) = [l_1,\dots,l_m]$, with $m = c_k - k$. Then*
(i) *For every $i = 1,\dots,m$ $l_i = \#[\Gamma(M{:}V_i)^+ \setminus \Gamma(M{:}V_{i-1})^+] \le r_i$;*
(ii) $\sum_{i=1}^m (r_i - l_i) \le \delta - k \le c - c_k$;
(iii) $\sum_{i=1}^m (r_i - l_i) = \delta - k$ *if and only if t.s.* $(M) = [r_1,\dots,r_m,1,\dots,1]$.

Proof. (i) Let $V_i^{(k)}{:=}\{x \in R_k,\ v(x) \ge s_i\},\ \forall i = 1,\dots,m$.
Then $V_i^{(k)} = V_i + t^{c_k}\bar{R}$ and $M{:}V_i^{(k)} = (M{:}V_i) \cap (M{:}t^{c_k}\bar{R}) = (M{:}V_i) \cap \bar{R}$. Hence $l_i = l_{R_k}(M{:}V_i^{(k)}/M{:}V_{i-1}^{(k)}) = l_R((M{:}V_i) \cap \bar{R}/(M{:}V_{i-1}) \cap \bar{R}) = \#[\Gamma(M{:}V_i)^+ \setminus \Gamma(M{:}V_{i-1})^+] \le \#[\Gamma(M{:}V_i) \setminus \Gamma(M{:}V_{i-1})] = r_i$ (recall Lemma 1.8).
(ii) and (iii) Combining Proposition 1.6(i) with Proposition 1.7(i), we obtain $c - c_k + \sum_{i=1}^m l_i = \sum_{i=1}^n r_i$. Hence, $\sum_{i=1}^m (r_i - l_i) = c - c_k - (\sum_{i=m+1}^n r_i) \le c - c_k - (n-m) = \delta - k$.
The second inequality of (ii) is obvious, since, by definition, $c_k - k \le c - \delta$. $\square$

To improve our analysis, it is useful to introduce the notion of type-sequence for the set $\Gamma(M)$. This is the natural extension of the definition of type-sequence for numerical semigroups, given in [1].

Again let $\Gamma = \{s_0 = 0 < s_1 < \cdots < s_n = c, \rightarrow\}$, $n = c - \delta$, be the value set of R. We consider, for every $i = 0,\dots,n$, the ideals $S_i{:=}\{s \in \Gamma, s \ge s_i\}$. Obviously, $S_n = [c, \rightarrow]$, $S_1 = \Gamma(\mathfrak{m})$, $S_0 = \Gamma$ and, in general, $S_i = \Gamma(V_i)$. Recall that $V_i{:=}\{x \in R,\ v(x) \ge s_i\}$.

From the maximal chain $S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_0$, we get the chain

$$\Gamma(M) - S_0 \subseteq \Gamma(M) - S_1 \subseteq \cdots \subseteq \Gamma(M) - S_n.$$

It is immediate that $\Gamma(M) - S_0 = \Gamma(M)$ and $\Gamma(M) - S_n = [c_k - c, \rightarrow]$. Let

$$\alpha_i{:=}\#[(\Gamma(M) - S_i) \setminus (\Gamma(M) - S_{i-1})], \quad i = 1,\dots,n.$$

Definition 1.11. The sequence $[\alpha_1,\dots,\alpha_n]$ is called the *type-sequence* of $\Gamma(M)$. We denote this by *t.s.* $(\Gamma_M){:=}[\alpha_1,\dots,\alpha_n]$.

Note now that $\Gamma(M)$ is also a Γ_k-set, where $\Gamma_k{:=}\Gamma(R_k)$.

We shall write $[\beta_1,\dots,\beta_m]$, $m = c_k - k$, to indicate the type-sequence of $\Gamma(M)$ regarded as a Γ_k*-set*, and denote it by *k-t.s.* $(\Gamma(M)){:=}[\beta_1,\dots,\beta_m]$.

Remark 1.12. (i) We have, by definition,

$$\beta_i = \#[(\Gamma(M) - S_i^{(k)}) \setminus (\Gamma(M) - S_{i-1}^{(k)})], \quad i = 1,\dots,m,$$

where

$$S_i^{(k)}:=\{s \in \Gamma_k, s \geq s_i\},$$

$$\alpha_1 = \#A(M), \quad \text{where} \quad A(M) = (\Gamma(M) - \Gamma(\mathfrak{m})) \setminus \Gamma(M),$$

$$\beta_1 = \#A_k(M), \quad \text{where} \quad A_k(M) = (\Gamma(M) - \Gamma(\mathfrak{m}_k)) \setminus \Gamma(M).$$

(ii) Since $\Gamma(M{:}\mathfrak{m}) \subseteq \Gamma(M) - \Gamma(\mathfrak{m})$, we obtain

$$\begin{array}{ccc} \beta_1 & \leq & \alpha_1 \\ | \vee & & | \vee \\ l_1 & \leq & r_1 \end{array}$$

We shall write $\alpha_1(M), \beta_1(M)$ instead of α_1, β_1, if there is the possibility of confusion. We can easily deduce the analogues of Propositions 1.6, 1.7 and 1.10.

Proposition 1.13. *Let $c(M) = c_k$, $m = c_k - k$, $n = c - \delta$. Then*
(i) $c - c_k + \delta(M) = \sum_{i=1}^n \alpha_i$;
(ii) $1 \leq \alpha_i \leq \alpha_1$, $\forall i = 1, \ldots, n$;
(iii) $\delta(M) = \sum_{i=1}^m \beta_i$;
(iv) $1 \leq \beta_i \leq \beta_1$, $\forall i = 1, \ldots, m$.

Proposition 1.14. *We have the following relations betweeen the α_i and β_i:*
(i) $\beta_i = \#[(\Gamma(M) - S_i)^+ \setminus (\Gamma(M) - S_{i-1})^+] \leq \alpha_i$, $\forall i = 1, \ldots, m$;
(ii) $\sum_{i=1}^m (\alpha_i - \beta_i) \leq \delta - k \leq c - c_k$;
(iii) $\sum_{i=1}^m (\alpha_i - \beta_i) = \delta - k$ *if and only if* t.s. $(\Gamma_M) = [\alpha_1, \ldots, \alpha_m, 1, \ldots, 1]$.

Proof of Propositions 1.13 and 1.14. The results follow by applying Propositions 1.6, 1.7 and 1.10 to the monomial module $M_0 = \sum_\gamma t^\gamma k$, $\gamma \in \Gamma(M)$, over the monomial ring $R_0 := \sum_\gamma t^\gamma k$, $\gamma \in \Gamma$. In fact, $\Gamma(M_0{:}V_i(M_0)) = \Gamma(M_0) - \Gamma(V_i(M_0)) = \Gamma(M) - S_i$. Hence $r_i(M_0) = \alpha_i$, $\forall i = 1, \ldots, n$, and $l_i(M_0) = \beta_i$, $\forall i = 1, \ldots, m$. $\square$

2.

This section is devoted to establishing suitable characterizations of modules having type-sequence of the form $[r, 1, \ldots, 1]$. Our study has been inspired by the papers of several authors, Barucci, D'Anna, Delfino, Dobbs, Fontana, Fröberg, who considered analogous properties in the case of rings (see [1–3]).

Canonical modules will play a crucial role in our context, so we begin this section by recalling Rosenlicht's definition of the canonical module and some of its basic properties.

The *dualizing module* of R is

$$\omega_R = \{w \in k\{\{t\}\}\,\mathrm{d}t \mid res(fw) = 0 \ \forall f \in R\}.$$

By means of the isomorphism $k\{\{t\}\}\,\mathrm{d}t \simeq k\{\{t\}\}$, which maps $\mathrm{d}t \mapsto 1$, we shall identify ω_R with a fractional ideal, again called ω_R.

Remark 2.1. The following properties of ω_R are well known (see [5]):
(i) $\omega_R{:}\omega_R = R$.
(ii) If $\mathfrak{a} \supseteq \mathfrak{b}$ are fractional ideals, then $\mathfrak{a} = \omega_R{:}(\omega_R{:}\mathfrak{a})$ and $l_R(\mathfrak{a}/\mathfrak{b}) = l_R(\omega_R{:}\mathfrak{b}/\omega_R{:}\mathfrak{a})$.
(iii) If S is an overring of R, S birational to R, then: $\omega_R{:}S = \omega_S$.

Notation 2.2. Once and for all we will fix the fractional ideal $\tilde{\omega}{:=}\varepsilon t^c \omega_R$ as the *canonical module* of R, where $\varepsilon \in \bar{R}$ is a unit chosen so that $R \subseteq \tilde{\omega} \subseteq \bar{R}$.

Remark 2.3. (i) Our choice for the canonical module $\tilde{\omega}$ implies that
(a) $c(\tilde{\omega}) = c$ and $\Gamma(\tilde{\omega}) = \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$ [6, Satz 5].
(b) R is Gorenstein if and only if $\tilde{\omega} = R$.
(ii) We note that, when R is not a Gorenstein ring, the choice of the unit ε is not unique. In fact, let $\tau \in \tilde{\omega} \setminus R$ with $v(\tau) > 0$; then $1+\tau \in \tilde{\omega} \setminus R$ and $R \subseteq (1+\tau)^{-1}\tilde{\omega} \subseteq \bar{R}$. Moreover, $(1+\tau)^{-1}\tilde{\omega} \neq \tilde{\omega}$, otherwise $(1+\tau) \in \tilde{\omega} : \tilde{\omega} = R$.
(iii) Recall that $r_1(M) = 1 \Leftrightarrow M \simeq \tilde{\omega}$, [5, 6.12], hence

$$t.s.\ (M) = [1,\dots,1] \Leftrightarrow M \simeq \tilde{\omega};$$
$$k\text{-}t.s.\ (M) = [1,\dots,1] \Leftrightarrow M \simeq \tilde{\omega}_k{:=}\tilde{\omega}_{R_k}.$$

For any fractional ideal N, it is possible to find $N' \simeq N$ and $N' \subseteq \tilde{\omega}$. This fact will be very useful later in formulating and proving our results. So, in the next proposition, we explain how to find such an N', which we will call an "*immersion of N in $\tilde{\omega}$*".

Proposition 2.4. *Let N be any fractional ideal.*
(i) *If $N \supseteq t^c\bar{R}$ and $\Gamma(N) \subseteq \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$, then there exists a unit $u \in \bar{R}$ such that $uN \subseteq \tilde{\omega}$.*
If $\Gamma(N) = \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$, then $uN = \tilde{\omega}$.
(ii) $\Gamma(\tilde{\omega}{:}N) = \Gamma(\tilde{\omega}) - \Gamma(N)$.
(iii) *If $N \subseteq \bar{R}$ and its conductor c_k verifies the condition $c_k - 1 \notin \Gamma$, then there exist units $u, u' \in \bar{R}$ such that $uN \subseteq \tilde{\omega}_k$ and $u' t^{c-c_k} N \subseteq \tilde{\omega}$.*

Proof. (i) is Lemma 1.1 of Oneto and Zatini [7], which is a slight generalization of Jäger [6, Satz 5].

(ii) The inclusion $\subseteq$ holds in general, so we have to prove the other one.

Choose $j \in \mathbb{Z}$ such that $j + \Gamma(N) \subseteq \Gamma(\tilde{\omega})$. Then $\Gamma(t^j N) \subseteq \Gamma(\tilde{\omega})$ and the module $N'{:=}t^j N + t^c\bar{R}$ satisfies the hypotheses of (i). Hence there exists some unit $u \in \bar{R}$ such that $ut^j N \subseteq \mu N' \subseteq \tilde{\omega} \Rightarrow ut^j \in \tilde{\omega}{:}N \Rightarrow j \in \Gamma(\tilde{\omega}{:}N)$.

(iii) is Proposition 1.2 of Oneto and Zatini [7]. □

The following examples show some of the misbehaviours of the immersions defined above:

Example 2.5. (1) Given a fractional ideal M, one can easily construct several immersions of M in $\tilde{\omega}$. To see this take, e.g., $M = R{:=}k[\![t^3,t^4,t^5]\!]$, hence $\tilde{\omega} = R + tR$. Here M and $(1+t)M$ are two immersions of M in $\tilde{\omega}$.

(2) Given M as above, it is not always possible to achieve both immersions of M in $\tilde{\omega}$ and in $\tilde{\omega}_k$ by means of the same unit. For instance consider the rings $R{:=}k[\![t^{15},t^{21},t^{25}+t^{28},t^{32}]\!]$ and $R_k{:=}k[\![t^{15},t^{21},t^{25}+t^{28},t^{29},\rightarrow]\!]$, $(k=25)$. Then one can compute that $\tilde{\omega} = \langle 1,t^{10},t^{11}(1-t^3),t^{13}(1-t^6),t^{17},t^{20}(1-t^3)\rangle R$ and $\tilde{\omega}_k = R_k + (1-t^3)^{-1}\langle t,t^2,t^4,t^5,t^6,t^8,t^9,t^{10},t^{11},t^{12},t^{14},t^{18}\rangle\, R_k$. Let $M{:=}\tilde{\omega}_k$ and suppose that there exists a unit $u \in \bar{R}$ such that $uM \subseteq \tilde{\omega}_k$ and $ut^{c-c_k}M \subseteq \tilde{\omega}$. Then $u \in \tilde{\omega}_k{:}\tilde{\omega}_k = R_k$, hence $ut^{c-c_k}M = t^{c-c_k}M \subseteq \tilde{\omega}$. Since $c=70$ and $c_k=29$, this would imply, in particular, that $t^{41} \in \tilde{\omega}$. If so, then $t^{69} = (t^{25}+t^{28})t^{41} - t^{30}(t^{25}+t^{28})t^{11}(1-t^3) - t^{72} \in \tilde{\omega}$, which is a contradiction.

(3) In general, there exists no unit $u \in \bar{R}$ such that $R \subseteq uM \subseteq \tilde{\omega}$.

Let $R{:=}k[\![t^5,t^8,t^{22}]\!]$ and $M{:=}(1+t^2)R+t^8R+t^{13}R$. In this case $c{=}20$ and $\tilde{\omega}{=}\langle 1,t^2\rangle R$. Suppose that $1 = um$, with $m \in M$ and u unit in $\bar{R}$, i.e., $1 = (1+b_1t+b_2t^2+b_3t^3+\cdots)(a_1+a_1t^2+a_2t^5+a_2t^7+a_3t^8+\cdots)$, $a_i,b_j \in k$. Then, by an easy calculation, we get $u = 1-t^2+t^4+\cdots$. Since $ut^{17} = t^{17}-t^{19}+t^{21}+\cdots \in uM$ and $t^{17} \in \tilde{\omega}$, the inclusion $uM \subseteq \tilde{\omega}$ would imply $c-1 = 19 \in \Gamma(\tilde{\omega})$, which is absurd.

Assumption 2.6. From now on, all *isomorphisms* will be multiplications by units of $\bar{R}$, so that isomorphic modules will have the same value set.

Given an overring S birational to R, property (iii) in Remark 2.1 allows us to find explicitly the immersion of the canonical module $\tilde{\omega}_S$ in $\tilde{\omega}$.

Lemma 2.7. *Let S be a ring such that $R \subseteq S \subseteq \bar{R}$ and let $c(S) = c_k$. Then*
(i) $\tilde{\omega}_S \simeq t^{c_k-c}\tilde{\omega}{:}S$.
(ii) *$\tilde{\omega}{:}S$ is the unique immersion of $\tilde{\omega}_S$ in $\tilde{\omega}$.*
(iii) *In particular*: $\tilde{\omega}_k \simeq t^{c_k-c}\tilde{\omega}{:}R_k = \bar{R} \cap t^{c_k-c}\tilde{\omega}$.

Proof. (i) Let $\tilde{\omega} = \varepsilon t^c \omega_R$, $\tilde{\omega}_S = \varepsilon' t^{c_k}\omega_S$, where ε,ε' are units of $\bar{R}$. Then

$$\tilde{\omega}_S = \varepsilon' t^{c_k}\omega_S = \varepsilon' t^{c_k}\omega_R{:}S = \varepsilon'\varepsilon^{-1}t^{c_k-c}\tilde{\omega}{:}S.$$

(ii) By (i) above: $\tilde{\omega}{:}S = vt^{c-c_k}\tilde{\omega}_S \subseteq \tilde{\omega}, v$ a unit of $\bar{R}$, hence $\tilde{\omega}{:}S$ is an immersion of $\tilde{\omega}_S$ in $\tilde{\omega}$ as in Proposition 2.4(iii). Moreover v is unique. In fact, given any other immersion $u't^{c-c_k}\tilde{\omega}_S \subseteq \tilde{\omega}$, we obtain:

$$u'v^{-1}\tilde{\omega}{:}S \subseteq \tilde{\omega}, \quad \text{i.e., } u'v^{-1} \in \tilde{\omega}{:}(\tilde{\omega}{:}S) = S = \tilde{\omega}_S{:}\tilde{\omega}_S.$$

Hence $u't^{c-c_k}\tilde{\omega}_S = vt^{c-c_k}\tilde{\omega}_S = \tilde{\omega}{:}S$.

(iii) The conclusion comes from (i), with $S = R_k = R + t^{c_k}\bar{R}$, since

$\tilde{\omega}{:}R_k = \tilde{\omega} \cap t^{c-c_k}\bar{R}$. □

In the next lemma we find another meaning for the number $\delta+\delta(M)-c_k$ (see Proposition 1.6(iii)). This will be useful in Theorem 2.10.

Lemma 2.8. *Let $uM\subseteq\tilde{\omega}_k$ and $u't^{c-c_k}M\subseteq\tilde{\omega}$ be immersions of M into the canonical modules of R_k and R, respectively. Call $M':=u't^{c-c_k}M$. Then*
(i) $\tilde{\omega}{:}M'\simeq\tilde{\omega}_k{:}M$;
(ii) $l_R(\tilde{\omega}/M')=\delta+\delta(M)-c_k$;
(iii) $l_R(\tilde{\omega}_k/uM)=k+\delta(M)-c_k$.

Proof. (i) By Proposition 2.7(iii) we have $\tilde{\omega}{:}t^{c-c_k}M=(\tilde{\omega}{:}t^{c-c_k}R_k){:}M\simeq\tilde{\omega}_k{:}M$.
(ii) Since $M'\supseteq t^c\bar{R}=u't^c\bar{R}$, we obtain the equalities

$$\begin{aligned}l_R(\tilde{\omega}/M')&=l_R(\tilde{\omega}/t^c\bar{R})-l_R(M'/u't^c\bar{R})=l_R(\bar{R}/R)-l_R(t^{c-c_k}M/t^c\bar{R})\\&=\delta-l_R(M/t^{c_k}\bar{R})=\delta+\delta(M)-c_k.\end{aligned}$$

(iii) Since $\delta(R_k)=k$, the assertion follows readily from (ii), regarding M as an R_k-module. □

Remark 2.9. Given any immersion $M':=u't^{c-c_k}M$ of M in $\tilde{\omega}$ we can see that

$$\begin{aligned}r_1(M)&=l_R(M'{:}\mathfrak{m}/M')\le l_R(\tilde{\omega}{:}\mathfrak{m}/M')\\&=l_R(\tilde{\omega}{:}\mathfrak{m}/\tilde{\omega})+l_R(\tilde{\omega}/M')=1+l_R(\tilde{\omega}/M').\end{aligned}\qquad(*)$$

Hence from statement (ii) above we can again deduce inequality in Proposition 1.6(iii).

Theorem 2.10 below shows that the R-modules which satisfy equality in $(*)$ are exactly those having type-sequence $[r_1(M),1,\ldots,1]$.

Theorem 2.10. *Let $R\subseteq M\subseteq\bar{R}, c(M)=c_k$, and let $M':=u't^{c-c_k}M\subseteq\tilde{\omega}$ be any immersion of M in $\tilde{\omega}$. The following conditions are equivalent*:
(i) $r_1(M)-1=\delta+\delta(M)-c(M)$;
(ii) $r_1(M)-1=l_R(\tilde{\omega}/M')$;
(iii) $\mathfrak{m}=\mathfrak{m}(\tilde{\omega}{:}M')$;
(iv) $\tilde{\omega}{:}\mathfrak{m}=M'{:}\mathfrak{m}$;
(v) $\mathfrak{m}\tilde{\omega}\subseteq M'$;
(vi) $t.s.(M)=[r_1(M),1,\ldots,1]$.

Proof. (i) ⇔ (ii): This is an immediate consequence of Lemma 2.8(ii).
(ii) ⇔ (iii): To establish this equivalence it is enough to observe that

$$\begin{aligned}l_R(\tilde{\omega}/M')&=l_R(\tilde{\omega}{:}M'/R)=l_R(\tilde{\omega}{:}M'/\mathfrak{m}(\tilde{\omega}{:}M'))+l_R(\mathfrak{m}(\tilde{\omega}{:}M')/\mathfrak{m})-1\\&=r_1(M')+l_R(\mathfrak{m}(\tilde{\omega}{:}M')/\mathfrak{m})-1.\end{aligned}$$

(iii) ⇔ (iv): It is trivial using properties (ii) of Remark 2.1, of the canonical module.
(iv) ⇔ (v): Note that, since $l_R(\tilde{\omega}{:}\mathfrak{m}/\tilde{\omega})=1$ and $c(\tilde{\omega})=c$, we have $\tilde{\omega}{:}\mathfrak{m}=\tilde{\omega}+t^{c-1}\bar{R}$. Then condition (iv) holds ⇔ $\tilde{\omega}{:}\mathfrak{m}\subseteq M'{:}\mathfrak{m}$ ⇔ $\tilde{\omega}\subseteq M'{:}\mathfrak{m}$ (because the inclusion $t^{c-1}\bar{R}\subseteq M'{:}\mathfrak{m}$ is always verified) ⇔ $\mathfrak{m}\tilde{\omega}\subseteq M'$.

(i) $\Leftrightarrow$ (vi): We know that $\sum_{i=2}^{n} r_i = c - c(M) + \delta(M) - r_1(M)$, where $n = c - \delta$. Hence, if $r_i = 1$, $\forall i = 2, \ldots, n$, then we obtain (i). Conversely, (i) implies that $\sum_{i=2}^{n} r_i = c - \delta - 1$ and, since $r_i \geq 1$, $i = 2, \ldots, n$, we are done. $\square$

Definition 2.11. We call any module M which satisfies the equivalent conditions of Theorem 2.10 "module having minimal type-sequence" (briefly an *m.t.s.* R-module). We call M *weakly m.t.s.* if it is *m.t.s.* as an R_k-module.

Remark 2.12. (i) From Proposition 1.10(i), we obtain that $r_i = 1 \Rightarrow l_i = 1$. Hence,

M is an m.t.s. module $\Rightarrow$ M is a weakly m.t.s. module.

(ii) Conditions (i)–(iv) of Theorem 2.10 can be viewed as the module theoretic analogue of the characterization of *almost Gorenstein* rings given in Proposition 20 of Barucci and Fröberg [2]. In particular, notice that, for $M = R$ relation (iii) becomes $\mathfrak{m} = \mathfrak{m}\tilde{\omega}$. More generally, for any overring $M \supseteq R$ having $c(M) = c$, (iii) becomes $\mathfrak{m} = \mathfrak{m}\tilde{\omega}_M$.

We have already introduced the notion of the type-sequence for the Γ-set $\Gamma(M)$, so it now seems natural to find an analogue of the *m.t.s.* property for this set and to compare the two properties. To do this, according to the terminology used in [2] for semigroups, we begin with the following:

Definition 2.13. We call $B_1 := \{c_k - 1 - x \mid x \in \Gamma\}$ the set of *holes of the first type* and $B_2 := \{x \in \mathbb{Z} \mid x \notin \Gamma(M)$ and $c_k - 1 - x \notin \Gamma\}$ the set of *holes of the second type*.

By definition we have that $B_1 \cap B_2 = \emptyset$. Moreover, $B(M) := B_1 \cup B_2 = \mathbb{N} \setminus \Gamma(M)$ is the complete set of holes of $\Gamma(M)$.

Proposition 2.14. *Using the notation above,* $A(M) = (\Gamma(M) - \Gamma(\mathfrak{m})) \setminus \Gamma(M)$, *let* $c(M) = c_k$. *Then*

(i) $B_2 = \Gamma(t^{c_k - c}\tilde{\omega}) \setminus \Gamma(M)$;
(ii) $B_2^+ = \{x \notin \Gamma(M) \mid c_k - 1 - x \notin \Gamma_k\} = \Gamma(\tilde{\omega}_k) \setminus \Gamma(M)$;
(iii) $\# B_2 = \delta - (c_k - \delta(M))$;
(iv) $\# B_2^+ = k - (c_k - \delta(M))$;
(v) $A(M) \subseteq B_2 \cup \{c_k - 1\}$.

Proof. (i) and (ii) follow directly from the definitions.

(iii) Using (i) and Remark 1.8 we get $\# B_2 = l_R(\tilde{\omega}/u' t^{c - c_k} M) = \delta + \delta(M) - c_k$, where the last equality is (ii) of Lemma 2.8.

(iv) Use (iii), regarding M as an R_k-module.

(v) Let $j \in A(M)$, $j \neq c_k - 1$; we have to prove that $c_k - 1 - j \notin \Gamma$. On the contrary, by definition of $A(M)$, $j + (c_k - 1 - j) = c_k - 1 \in \Gamma(M)$. This is a contradiction. $\square$

Looking at valuations, we obtain the analogue of Theorem 2.10.

Theorem 2.15. *The following are equivalent*:
(i) $A(M)=B_2\cup\{c_k-1\}$;
(ii) $\alpha_1(M)-1=\delta+\delta(M)-c_k$;
(iii) $\Gamma(\mathfrak{m})=\Gamma(\mathfrak{m})+\Gamma(\tilde{\omega}{:}t^{c-c_k}M)$;
(iv) *t.s.* $(\Gamma(M))=[\alpha_1,1,\ldots,1]$.

Proof. (i) $\Leftrightarrow$ (ii): It follows from Proposition 2.14.

(i) $\Rightarrow$ (iii): We have $B_2\subseteq A(M)\Rightarrow\Gamma(t^{c_k-c}\tilde{\omega})+\Gamma(\mathfrak{m})\subseteq\Gamma(M)\Rightarrow\Gamma(\tilde{\omega})+\Gamma(\mathfrak{m})+\Gamma(\tilde{\omega}{:}t^{c-c_k}M)\subseteq\Gamma(t^{c-c_k}M)+\Gamma(\tilde{\omega}{:}t^{c-c_k}M)\subseteq\Gamma(\tilde{\omega})\Rightarrow\Gamma(\mathfrak{m})+\Gamma(\tilde{\omega}{:}t^{c-c_k}M)\subseteq\Gamma(\tilde{\omega})-\Gamma(\tilde{\omega})=\Gamma$. From this, since $\Gamma(\tilde{\omega}{:}t^{c-c_k}M)\subseteq\mathbb{N}$, by (i) of Lemma 2.8, we can deduce that $\Gamma(\mathfrak{m})+\Gamma(\tilde{\omega}{:}t^{c-c_k}M)\subseteq\Gamma(\mathfrak{m})$.

(iii) $\Rightarrow$ (i): By (v) of Proposition 2.14, it suffices to show the inclusion $B_2\cup\{c_k-1\}\subseteq A(M)$. Let $j\in B_2\Rightarrow j\notin\Gamma(M)$ and $j=c_k-c+x$, $x\in\Gamma(\tilde{\omega})$.

Using the hypothesis, $x+\Gamma(\mathfrak{m})\subseteq\Gamma(\tilde{\omega})\Rightarrow x+\Gamma(\mathfrak{m})+\Gamma(\tilde{\omega}{:}t^{c-c_k}M)\subseteq\Gamma(\tilde{\omega})$, i.e., $j+\Gamma(\mathfrak{m})+\Gamma(\tilde{\omega}{:}M)\subseteq\Gamma(\tilde{\omega})\Rightarrow j+\Gamma(\mathfrak{m})\subseteq\Gamma(M)$, by Proposition 2.4(ii). Hence $j\in A(M)$.

(ii) $\Leftrightarrow$ (iv): By Proposition 1.13 $c-c_k+\delta(M)=\alpha_1+\sum_{i=2}^n\alpha_i$. Hence the hypothesis $\alpha_i=1\ \forall i=2,\ldots,n$ implies (ii). Conversely, if (ii) holds, then $\sum_{i=2}^n\alpha_i=c-\delta-1$ and since $\alpha_i\geq 1, i=2,\ldots,n$, claim (iv) is proved. $\square$

Definition 2.16. We call $\Gamma(M)$ *m.t.s.* if M fulfills the equivalent conditions of Theorem 2.15.

We call $\Gamma(M)$ *weakly m.t.s.* if M fulfills these conditions as an R_k-module.

Remark 2.17. By analogy with Remark 2.12(i), and using Proposition 1.14(ii), we obtain that

$$\Gamma(M)\text{ is m.t.s.}\ \Rightarrow\ \Gamma(M)\text{ is weakly m.t.s.}$$

The link between the two cases, M *m.t.s.* and $\Gamma(M)$ *m.t.s.*, is very strong.

Theorem 2.18. (i) *M is m.t.s. $\Leftrightarrow$ $\Gamma(M)$ is m.t.s. and $r_1(M)=\alpha_1(M)$*;
(ii) *M is weakly m.t.s. $\Leftrightarrow$ $\Gamma(M)$ is weakly m.t.s. and $l_1(M)=\beta_1(M)$.*

Proof. Both implications: $\Leftarrow$ follow directly from the definitions.

(i) $\Rightarrow$: By (iii) and (v) of Proposition 2.14, the following inequalities hold:

$$r_1(M)-1\leq\alpha_1(M)-1\leq\#B_2=\delta-(c_k-\delta(M)),$$

which become equalities in the *m.t.s.* case.

(ii) $\Rightarrow$: Similarly, we have the following inequalities:

$$l_1(M)-1\leq\beta_1(M)-1\leq\#B_2^+=k-(c_k-\delta(M)),$$

which become equalities in the weakly *m.t.s.* case. $\square$

Corollary 2.19. *M is m.t.s. $\Leftrightarrow$ M is weakly m.t.s. and $r_1(M)-l_1(M)=\delta-k$.*

Proof. $\Rightarrow$: As we noticed, M is *m.t.s.* $\Rightarrow M$ is *weakly m.t.s*. So, by definition, $r_1(M) - l_1(M) = \delta - k$.

$\Leftarrow$: To get this implication, we simply recall Proposition 1.10(iii). $\square$

In the last part of this section, we want to characterize the subclass of *m.t.s.* R-modules of CM-type 2.

Theorem 2.20. *Let* $c(M) = c_k$. *The following statements are equivalent*:
(i) *M is m.t.s. and* $r_1(M) = 2$;
(ii) *for any immersion of M in* $\tilde{\omega}$, $l_R(\tilde{\omega}/u't^{c-c_k}M) = 1$;
(iii) $\delta + \delta(M) - c_k = 1$;
(iv) $t.s.(M) = [2, 1, \ldots, 1]$;
(v) *$\Gamma(M)$ is m.t.s. and* $\alpha_1(M) = 2$;
(vi) $t.s.(\Gamma(M)) = [2, 1, \ldots, 1]$.

Proof. (i) $\Rightarrow$ (ii): It is true by definition of *m.t.s.*

(ii) $\Rightarrow$ (i): In general, we have (see Proposition 1.6(iii)) $r_1(M) - 1 \le \delta + \delta(M) - c_k$; then $r_1(M) \le 2$. But $r_1(M) = 1$ implies $M \simeq \tilde{\omega}$ [5, Korollar 6.12], hence $\delta + \delta(M) - c_k = 0$ (see, e.g., Proposition 1.5 of Oneto and Zatini [9]), against our hypothesis. Therefore $r_1(M) = 2$ and M is *m.t.s.*
(ii) $\Leftrightarrow$ (iii): It follows from Lemma 2.8(ii).
(i) $\Leftrightarrow$ (iv): By Theorem 2.10.
(i) $\Leftrightarrow$ (v): By Theorem 2.18, since $r_1(M) \le \alpha_1(M)$.
(v) $\Leftrightarrow$ (vi): By Theorem 2.15. $\square$

Definition 2.21. We call M *almost canonical* if it fulfills the equivalent conditions of Theorem 2.20.

Remark 2.22. (i) The name "almost canonical" is motivated by the fact that these modules can be easily constructed by deleting one element in a minimal system of generators of the canonical module $\tilde{\omega}$.

(ii) A ring R is said to be a *Kunz ring* if it satisfies the equivalent conditions of Theorem 2.20 (see [1]).

Corollary 2.23. *Let S be any overring*, $R \subseteq S \subseteq \bar{R}$. *Then $\tilde{\omega}_S$ is almost canonical as an R-module* $\Leftrightarrow l_R(S/R) = 1$.

Proof. Just apply Theorem 2.20, part (ii), and Lemma 2.7(ii) to $M = \tilde{\omega}_S$. Then $\tilde{\omega}_S$ is almost canonical if and only if $1 = l_R(\tilde{\omega}/\tilde{\omega}{:}S) = l_R(S/R)$. $\square$

There are only a few modules, besides the ones considered in Corollary 2.23, which are almost canonical, as the following theorem shows.

Theorem 2.24. *M is almost canonical if and only if*
(i) $k=\delta-1$ *and* $M\simeq\tilde{\omega}_k$ or
(ii) $k=\delta$ *and* $l_R(\tilde{\omega}/uM)=1$, *for any immersion* $uM\subseteq\tilde{\omega}$.
In this last case there are two possibilities:
(a) $M{:}M=R$;
(b) $M\simeq\tilde{\omega}_{M:M}$ *and* $l_R(M{:}M/R)=1$.

Proof. Suppose M is almost canonical, then $l_1(M)\le r_1(M)=2$. If $l_1(M)=1$, then $M\simeq\tilde{\omega}_k$ and $\delta-k=r_1(M)-l_1(M)=1$ by 2.19. If $l_1(M)=2$, then $\delta-k=0$, i.e., $c(M)=c$ and by (ii) of Theorem 2.20 $l_R(\tilde{\omega}/uM)=1$. Vice versa, if $k=\delta-1$ and $M\simeq\tilde{\omega}_k$, then $l_R(R_k/R)=1$. So, by Corollary 2.23, M is almost canonical. The same holds if $k=\delta$ and $l_R(\tilde{\omega}/uM)=1$, by Theorem 2.20(ii).

To complete the proof we must show that if M is almost canonical and $c(M)=c$, i.e., $k=\delta$, then either $M{:}M=R$ or $M\simeq\tilde{\omega}_{M:M}$. We know that

$$c_k-\delta(M)\le\delta(M{:}M)\le\delta\quad\text{(see [7, Proposition 2.1(ii)]).}$$

From our hypotheses it follows that $\delta-1\le\delta(M{:}M)\le\delta$.

Hence either $\delta(M{:}M)=\delta$ or $\delta(M{:}M)=\delta-1$.

In the first case $M{:}M=R$ and we are done.

Now let $\delta(M{:}M)=\delta-1$. We have $c(\tilde{\omega}_{M:M})=\delta(M{:}M)+\delta(\tilde{\omega}_{M:M})$, by Proposition 1.5 of Oneto and Zatini [9] and also $c(\tilde{\omega}_{M:M})=c(M)$: using the hypothesis $c(M)=\delta(M)+\delta-1$, we conclude that $\delta(M)=\delta(\tilde{\omega}_{M:M})$. Hence $M\simeq\tilde{\omega}_{M:M}$, because there exists an immersion $vM\subseteq\tilde{\omega}_{M:M}$. Finally, by Corollary 2.23, $l_R(M{:}M/R)=1$. □

Example 2.25. Let $R:=k[\![t^5,t^{12},t^{13},t^{14},t^{16}]\!]$.

1. Let $M:=R+t^2R+t^3R+t^9R$. We see easily that $\Gamma(R)=\{0,5,10,12,\rightarrow\}$ and $\Gamma(M)=\{0,2,3,5,7,8,9,10,12\rightarrow\}$. Hence M is almost canonical and $M=\tilde{\omega}_{M:M}$.
2. Let $M:=R+tR+t^2R+t^3R$. We have: $\Gamma(M)=\{0,1,2,3,5,6,7,8,10,\rightarrow\}$ and $A(M)=\{-2,7\}$. Thus $k=8$, $r_1(M)=2, l_1(M)=1$, and $\delta+\delta(M)-c(M)=9+2-10=1$. So M is almost canonical and $M\simeq\tilde{\omega}_k$.
3. Let $M:=R+t^2R+t^3R+t^7R+t^9R+t^{12}R$. Here $\Gamma(M)=\{0,2,3,5,7,\rightarrow\}$, $k=5$, $A(M)=\{-5,-3,-2,4,6\}$. Hence $r_1(M)=5, l_1(M)=2$. In this case, since $k+\delta(M)-c(M)=5+3-7=1$, M is almost canonical over R_k, i.e., k-*t.s.* $(M)=[2,1]$. But, since $\delta+\delta(M)-c(M)=9+3-7\ne4$, M is not *m.t.s.* over R. According to Theorem 2.24, in this case we have $M{:}M=R_k$.

3.

In Theorem 2.10, (v), the *m.t.s.* property of R-modules was characterized by means of the R-module $\mathfrak{m}\tilde{\omega}$, which is isomorphic to the canonical module of the ring $\mathfrak{m}{:}\mathfrak{m}$ (for this isomorphism see Proposition 3.1(iii) below). This fact naturally suggests that one studies the connection between the ring $\mathfrak{m}{:}\mathfrak{m}$ and the *m.t.s.* property. This is done in the first part of the present section.

In the last part of the section, various characterizations of *m.t.s.* modules and of reflexive modules over an almost Gorenstein ring are given.

We begin by considering the ring $R' := \mathfrak{m}{:}\mathfrak{m}$. Of course we have $R \subseteq R' \subseteq \bar{R}$. According to the notation used in the previous sections, $c(R')$ will denote the conductor of R' in $\bar{R}$, $\omega_{R'}$ and $\tilde{\omega}_{R'}$ the dualizing module and the canonical module of R' respectively, and $\delta(R')$ the singularity degree of R'.

Proposition 3.1. *Let* $R' := \mathfrak{m}{:}\mathfrak{m}$ *and let* $\tilde{\omega} = \varepsilon t^c \omega_R$, *where* $\varepsilon \in \bar{R}$ *is the unit fixed in* 2.2. *Then*
(i) $c(R') = c - e$ *and* $\delta(R') = \delta - r_1(R)$;
(ii) $\omega_{R'} = \mathfrak{m}\omega_R$;
(iii) $R' \subseteq \varepsilon t^{c-e}\omega_{R'} \subseteq \bar{R}$, *i.e.* $\varepsilon t^{c-e}\omega_{R'}$ *is a canonical module of* R'.

Proof. (i) The two equalities follow easily from the definitions.
(ii) Since we assume $e > 1$, then $R{:}\mathfrak{m} = \mathfrak{m}{:}\mathfrak{m}$, and by duality we get $\omega_{R'} = \omega_R{:}R' = \omega_R{:}(\omega_R{:}\mathfrak{m}\omega_R) = \mathfrak{m}\omega_R$.
(iii) We have $R' \subseteq t^{-e}\mathfrak{m} \subseteq t^{-e}\mathfrak{m}\tilde{\omega} = \varepsilon t^{c-e}\mathfrak{m}\omega_R = \varepsilon t^{c-e}\omega_{R'} \subseteq \bar{R}$. $\square$

Statement (iii) above says that the same unit allows us to immerse both the canonical modules of R and R' in $\bar{R}$, in such a way so as to verify the condition of Assumption 1.1.

Notation 3.2. We fix as the canonical module of R' the module $\tilde{\omega}_{R'} := \varepsilon t^{c-e}\omega_{R'}$.

Proposition 3.3. *The following facts hold*:
(i) $\tilde{\omega}_{R'} = t^{-e}\mathfrak{m}\tilde{\omega}$;
(ii) $\mathfrak{m}\tilde{\omega}$ *is the* (*unique*) *immersion of* $\tilde{\omega}_{R'}$ *in* $\tilde{\omega}$;
(iii) $\tilde{\omega}_{R'}$ *is a m.t.s. R-module.*

Proof. (i) is immediate by (ii) of Proposition 3.1: $\tilde{\omega}_{R'} = \varepsilon t^{c-e}\omega_{R'} = \varepsilon t^{c-e}\mathfrak{m}\omega_R = t^{-e}\mathfrak{m}\tilde{\omega}$.
(ii) follows readily from Lemma 2.7, (ii), since $\tilde{\omega}{:}R' = \mathfrak{m}\tilde{\omega}$.
(iii) Condition (v) of Theorem 2.10 is trivially satisfied if $M = \tilde{\omega}_{R'}$. $\square$

More generally,

Proposition 3.4. *Let* S *be any overring,* $R \subseteq S \subseteq \bar{R}$ *and let* $c(S) = c_k$. *Then*
(i) $\tilde{\omega}_S$ *is an m.t.s. R-module* $\Leftrightarrow S \subseteq \mathfrak{m}{:}\mathfrak{m}$;
(ii) $\tilde{\omega}_k$ *is an m.t.s. R-module* $\Leftrightarrow c_k \geq c - e$.

Proof. (i) By (v) of Theorem 2.10, applied with $M = \tilde{\omega}_S$, and by Lemma 2.7(ii), we get $\tilde{\omega}_S$ is *m.t.s.* $\Leftrightarrow \mathfrak{m}\tilde{\omega} \subseteq \tilde{\omega}{:}S$. This, by properties 2.1, is equivalent to $S \subseteq \mathfrak{m}{:}\mathfrak{m}$.

(ii) follows readily from (i). In fact $R_k \subseteq \mathfrak{m}{:}\mathfrak{m} \Leftrightarrow t^{c_k}\bar{R} \subseteq \mathfrak{m}{:}\mathfrak{m} \Leftrightarrow c_k \geq c - e$.

Proposition 3.5. *The following conditions are equivalent*:
(i) $\mathfrak{m}{:}\mathfrak{m}$ *is an m.t.s. R-module*;
(ii) R *is almost Gorenstein and* $r_1(R) = e - 1$;
(iii) $\mathfrak{m}{:}\mathfrak{m}$ *is a Gorenstein ring*.

Proof. First observe that $t^e(\mathfrak{m}{:}\mathfrak{m}) \subseteq \mathfrak{m} \subseteq \mathfrak{m}\tilde{\omega}$. Moreover

$$r_1(R) = e - 1 \Leftrightarrow l_R(t^e(\mathfrak{m}{:}\mathfrak{m})/t^e\mathfrak{m}) = e \Leftrightarrow t^e(\mathfrak{m}{:}\mathfrak{m}) = \mathfrak{m}.$$

Now apply Theorem 2.10, (v), to the R-module $\mathfrak{m}{:}\mathfrak{m}$. It is *m.t.s.* if and only if $\mathfrak{m}\tilde{\omega} = t^e(\mathfrak{m}{:}\mathfrak{m})$ (in this case $c_k = c - e$ and $u' = 1$).

(i) $\Leftrightarrow$ (ii): It follows immediately from these observations.

(ii) $\Leftrightarrow$ (iii): The ring $\mathfrak{m} : \mathfrak{m}$ is Gorenstein if and only if $\mathfrak{m}{:}\mathfrak{m} = \tilde{\omega}_{\mathfrak{m}:\mathfrak{m}}$. This happens, by Proposition 3.1, (iii), if and only if $t^e(\mathfrak{m}{:}\mathfrak{m}) = \mathfrak{m}\tilde{\omega}$. $\square$

A more general version of statement (ii) $\Leftrightarrow$ (iii) is given in [2, Proposition 25], for a one-dimensional local CM-ring R with finite closure and with a canonical ideal $\tilde{\omega}$ such that $R \subseteq \tilde{\omega} \subseteq \bar{R}$.

The next proposition involves the overring $M{:}M$ and its δ-invariant $\delta(M{:}M)$, which has an interesting geometric meaning. Let $\mathscr{M}(R)$ be the reduced variety, constructed by Greuel and Pfister [4], which parametrizes (up to isomorphism) all finitely generated torsion-free R-modules of rank 1. The isomorphism classes of such modules correspond to orbits under the action on $\mathscr{M}(R)$ of the group $(\bar{R}/t^{2\delta}\bar{R})^*/k^*$. The dimension of the orbit for a given M is exactly $\delta(M{:}M)$. (Some properties of this invariant are studied also in [7].)

Proposition 3.6. *Let M be m.t.s. Then*
(i) $M{:}M \subseteq \mathfrak{m}{:}\mathfrak{m}$;
(ii) $\delta - r_1(R) \leq \delta(M{:}M) \leq k$;
(iii) $c_k \geq c - e$;
(iv) $r_1(M) \leq r_1(R) + 1$.
Moreover, denoting by M' *an immersion of* M *in* $\tilde{\omega}$, *the following are equivalent*:
(v) $M{:}M = \mathfrak{m}{:}\mathfrak{m}$;
(vi) $\delta(M{:}M) = \delta - r_1(R)$;
(vii) $r_1(M) = r_1(R) + 1$;
(viii) $M \simeq \tilde{\omega}_{\mathfrak{m}:\mathfrak{m}}$;
(ix) $\mathfrak{m}\tilde{\omega} = M'$.

Proof. (i) We have obviously that $M{:}M = M'{:}M' \subseteq \tilde{\omega} : M'$ and (iii) of Theorem 2.10 holds by hypothesis; so $\mathfrak{m}(M{:}M) \subseteq \mathfrak{m}(\tilde{\omega}{:}M') = \mathfrak{m}$.

(ii) It suffices to consider the inclusions $R_k \subseteq M{:}M \subseteq \mathfrak{m}{:}\mathfrak{m}$ and to compute the invariants $\delta()$.

(iii) follows immediately from (i).

The hypothesis M *m.t.s.* implies that $\mathfrak{m}\tilde{\omega} \subseteq M'$ and that $l_R(\tilde{\omega}/M') = r_1(M) - 1$. Then $r_1(R) = l_R(\tilde{\omega}/\mathfrak{m}\tilde{\omega}) = l_R(\tilde{\omega}/M') + l_R(M'/\mathfrak{m}\tilde{\omega}) = r_1(M) - 1 + l_R(M'/\mathfrak{m}\tilde{\omega})$.

From this we deduce (iv) and the equivalence (vii) $\Leftrightarrow$ (ix).

(v) $\Leftrightarrow$ (vi): It is obvious since, by (i), $M{:}M \subseteq \mathfrak{m}{:}\mathfrak{m}$.

(vi) $\Rightarrow$ (vii): By (iv) $r_1(M) \leq r_1(R)+1$, so we have only to prove $r_1(M) \geq r_1(R)+1$. We have, in general, that: $c_k - \delta(M) \leq \delta(M{:}M)$ (see [7, Proposition 2.1 (ii)]) and, by virtue of the *m.t.s.* property of M, we obtain $r_1(M) - 1 = \delta + \delta(M) - c_k$. Thus $r_1(M) = 1 + \delta - (c_k - \delta(M)) \geq 1 + \delta - \delta(M{:}M) = r_1(R) + 1$.

(viii) $\Leftrightarrow$ (ix): It follows easily from Proposition 3.3(i).

(viii) $\Rightarrow$ (vi): By hypothesis $M{:}M = \mathfrak{m}{:}\mathfrak{m}$ and $\delta(\mathfrak{m}{:}\mathfrak{m}) = \delta - r_1(R)$. $\square$

Remark 3.7. Proposition 3.4 allows us to construct examples of *m.t.s.* R-modules with conductor $c - e$, but not isomorphic to $\tilde{\omega}_{R'}$, where $R' = \mathfrak{m}{:}\mathfrak{m}$. Take, for instance, R such that there exists a ring $S \neq R'$, $R \subseteq S \subseteq R'$, with $c(S) = c - e$. Then the canonical module $\tilde{\omega}_S$ has the required features.

Proposition 3.8. *The following conditions are equivalent*:

(i) *R is almost Gorenstein*;

(ii) *Every R-module M having $c(M) = c$ is m.t.s*;

(iii) $r_1(R) = l_R(M/R) + r_1(M)$ *for all M such that* $c(M) = c$;

(iv) $r_1(R) = l_R(\tilde{\omega}/R) + r_1(\tilde{\omega})$.

Proof. Recall that R is almost Gorenstein if and only if $\mathfrak{m} = \mathfrak{m}\tilde{\omega}$.

(i) $\Rightarrow$ (ii): Let M be an R-module such that $c(M) = c$ and let $uM \subseteq \tilde{\omega}$ be any immersion of M in $\tilde{\omega}$. We claim that: $\mathfrak{m} = \mathfrak{m}(\tilde{\omega}{:}uM)$. In fact we have: $u\mathfrak{m} \subseteq \mathfrak{m}(\tilde{\omega}{:}M) \subseteq \mathfrak{m}\tilde{\omega} = \mathfrak{m}$. Since u is a unit, the above inclusions are equalities.

(ii) $\Rightarrow$ (i): It is obvious.

(i) $\Rightarrow$ (iii): By the proof of (i) $\Rightarrow$ (ii), we have that $u\mathfrak{m} = \mathfrak{m} = \mathfrak{m}(\tilde{\omega}{:}M)$. Then

$$\begin{aligned} r_1(R) &= l_R(\tilde{\omega}/\mathfrak{m}\tilde{\omega}) = l_R(\tilde{\omega}/\mathfrak{m}) = l_R(\tilde{\omega}/\tilde{\omega}{:}M) + l_R(\tilde{\omega}{:}M/\mathfrak{m}) \\ &= l_R(M/R) + l_R(\tilde{\omega}{:}M/\mathfrak{m}(\tilde{\omega}{:}M)) = l_R(M/R) + r_1(M). \end{aligned}$$

(iii) $\Rightarrow$ (iv): It is obvious.

(iv) $\Leftrightarrow$ (i): See Proposition 20 of [2]. $\square$

Example 3.9. This example shows that if R is almost Gorenstein, then R_k need not be almost Gorenstein.

Let $R{:=}k[\![t^4, t^9, t^{14}, t^{19}]\!]$. Since $\Gamma(R) = \{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}$, $\delta = 9$, $c = 16$, $r_1(R) = 3$, then R is an almost Gorenstein ring. Take $k = 8$, so that $\Gamma(R_k) = \{0, 4, 8, 9, 12, \rightarrow\}$, $\delta(R_k) = 8$, $c(R_k) = 12$ and $r_1(R_k) = 3$, hence R_k is not almost Gorenstein.

Corollary 3.10. *Suppose that R is almost Gorenstein and that $r_1(R)=2$. Then there are exactly two isomorphism classes of R-modules whose conductor is c: the class of R and the class of $\tilde{\omega}$.*

Proof. The statement follows immediately from assertion (iii) of the above proposition, since $r_1(R)=2$. □

We conclude this section by comparing the reflexiveness and the *m.t.s.* property for a fractional ideal M. We recall that, in our hypotheses, M is reflexive if and only if $M=M^{**}$, where $M^{**}:=R:(R:M)$.

A slight generalization of Proposition 22 of [2], to modules, is the following:

Proposition 3.11. *Let M be an R-module and let $c(M)=c_k$ (recall Assumption 1.1 and Notation 1.3). Then*

$$l_R(M/R) \leq l_R(R/R{:}M) + l_R(\tilde{\omega}/R). \qquad (*)$$

Consider the conditions
(i) $\tilde{\omega} \subseteq M{:}M$, *i.e.*, $\tilde{\omega}M=M$;
(ii) $R{:}M=(R{:}M)\tilde{\omega}=\tilde{\omega}{:}M$;
(iii) $=$ *holds in* $(*)$;
(iv) $M=M^{**}$;
(v) $(R{:}M)\tilde{\omega}=\tilde{\omega}{:}M$.
Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v).
If R is almost Gorenstein and $M \neq R$, then all conditions are equivalent.

Proof. To prove $(*)$ it suffices to note that
$l_R(M/R)=l_R(\tilde{\omega}/\tilde{\omega}{:}M) \leq l_R(\tilde{\omega}/R{:}M)=l_R(\tilde{\omega}M/R)=l_R(R/R{:}M)+l_R(\tilde{\omega}/R)$. Then the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are clear. Moreover, by Remark 2.1: $l_R(M^{**}/M)=l_R(\tilde{\omega}{:}M/(R{:}M)\tilde{\omega})$, so we get (iv) $\Leftrightarrow$ (v).

(ii) $\Rightarrow$ (v) is trivial. Suppose now that R is almost Gorenstein and $M \neq R$. Since $R{:}M=\mathfrak{m}{:}M$ and $\mathfrak{m}=\mathfrak{m}\tilde{\omega}$, we get $(R{:}M)M\tilde{\omega} \subseteq \mathfrak{m}$. Hence $(R{:}M)\tilde{\omega}=R{:}M$. So, (v) $\Rightarrow$ (ii). □

The following example shows that the hypothesis "R is almost Gorenstein" is necessary to state the equivalence of the conditions in Proposition 3.11:

Example 3.12. Let $R:=k[\![t^3,t^7,t^8]\!]$, $M:=R+t^4R+t^5R$. Then $M=M^{**}$, but $\tilde{\omega}M \neq M$, because $t \in \tilde{\omega}M$.

Remark 3.13. (i) If $M=M^{**}$ and $M \neq R$, then $R{:}M \subseteq \mathfrak{m}$ implies that $R{:}\mathfrak{m} \subseteq M^{**}=M$. Hence $c(M) \leq c-e$. In particular, the only reflexive R-module M such that $R \subseteq M \subseteq \bar{R}$ and $c(M)=c$ is $M=R$.

(ii) If M is reflexive, then $M{:}M$ is reflexive. In fact, $M{:}M = R{:}(R{:}M)M$ and $R{:}I$ is reflexive for every non-zero fractional R-ideal I.

The converse of the statement of Remark 3.13, (ii) characterizes almost Gorenstein rings.

Proposition 3.14. *The following conditions are equivalent*:
(i) *R is almost Gorenstein*;
(ii) *For every R-module M, $M \neq R$, the following holds*:
M is reflexive $\Leftrightarrow$ $M{:}M$ is reflexive and $M{:}M \neq R$;
(iii) *$\mathfrak{m}\tilde{\omega}$ is reflexive.*

Proof. (i) $\Rightarrow$ (ii): Let M be an R-module, $M \neq R$. Put $S{:=}M{:}M$, and apply the equivalence (iv) $\Leftrightarrow$ (i) of Proposition 3.11 to M and S. Then M is reflexive $\Leftrightarrow \tilde{\omega} \subseteq S = S : S \Leftrightarrow S \neq R$ and S is reflexive.

(ii) $\Rightarrow$ (iii): The module $M = \mathfrak{m}\tilde{\omega}$ is isomorphic to the canonical module of R' (see Proposition 3.3(ii)). Hence $\mathfrak{m}\tilde{\omega}{:}\mathfrak{m}\tilde{\omega} = R' = R{:}\mathfrak{m}$ is reflexive.

(iii) $\Rightarrow$ (i): The hypothesis "$\mathfrak{m}\tilde{\omega}$ reflexive" and the equivalence (iv) $\Leftrightarrow$ (v) of Proposition 3.11 give $(R{:}\mathfrak{m}\tilde{\omega})\tilde{\omega} = \tilde{\omega}{:}\mathfrak{m}\tilde{\omega} = R{:}\mathfrak{m}$. From the last equality, we deduce that $0 \in \Gamma(R{:}\mathfrak{m}\tilde{\omega}) \subseteq \mathbb{N}$ and that $(R{:}\mathfrak{m}\tilde{\omega})\mathfrak{m}\tilde{\omega} = \mathfrak{m} \subseteq \mathfrak{m}\tilde{\omega}$. Hence, by comparing valuations, we obtain that $\mathfrak{m} = \mathfrak{m}\tilde{\omega}$, i.e., R is almost Gorenstein.

Corollary 3.15. *Let R be almost Gorenstein and let M be an R-module, $M \neq R$. Then M is reflexive $\Leftrightarrow$ $M{:}M \supseteq \mathfrak{m}{:}\mathfrak{m}$*

Proof. The implication $\Rightarrow$ always holds.

The implication $\Leftarrow$ follows from (ii) of Proposition 3.14, since, by Proposition 22 of Barucci and Fröberg [2], $M{:}M$ is reflexive.

Corollary 3.16. *Suppose R is not Gorenstein and let $M \neq R$. Then M is m.t.s. and reflexive $\Leftrightarrow$ R is almost Gorenstein and $M \simeq \tilde{\omega}_{\mathfrak{m}:\mathfrak{m}}$.*

Proof. We have to prove $\Rightarrow$, since Propositions 3.14 and 3.4(i) give the converse. If M is *m.t.s.* and reflexive, we obtain that $M{:}M = \mathfrak{m}{:}\mathfrak{m}$ (hence also $c(M) = c - e$). We show the two inclusions.

$M{:}M \supseteq \mathfrak{m}{:}\mathfrak{m}$. The hypothesis $M \neq R$ implies that $(R{:}M)M \subseteq \mathfrak{m}$. Since M is reflexive, then $M{:}M = R{:}(R{:}M)M \supseteq R{:}\mathfrak{m}$.

$M{:}M \subseteq \mathfrak{m}{:}\mathfrak{m}$. This is immediate by Proposition 3.6(i), since M is *m.t.s.*

Applying Proposition 3.6(v) $\Rightarrow$ (viii), we deduce that $M \simeq \tilde{\omega}_{\mathfrak{m}:\mathfrak{m}} = t^{-e}\mathfrak{m}\tilde{\omega}$. But, by the above proposition, $\mathfrak{m}\tilde{\omega}$ reflexive means that R is almost Gorenstein.

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